

BLOCK ALGEBRA IN TWO-COMPONENT BKP AND D TYPE DRINFELD-SOKOLOV HIERARCHIES

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ABSTRACT. We construct generalized additional symmetries of a two-component BKP hierarchy defined by two pseudo-differential Lax operators. These additional symmetry flows form a Block type algebra with some modified (or additional) terms because of a B type reduction condition of this integrable hierarchy. Further we show that the D type Drinfeld-Sokolov hierarchy, which is a reduction of the two-component BKP hierarchy, possess a complete Block type additional symmetry algebra. That D type Drinfeld-Sokolov hierarchy has a similar algebraic structure as the bigraded Toda hierarchy which is a differential-discrete integrable system.

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1. INTRODUCTION

One interesting topic in the study of integrable hierarchies is to find symmetry and its recursion relation, and further to identify its algebraic structures. There are already many results in literatures, for example [1]-[6]. Among these symmetries, the additional symmetry is a relatively new type and has been studied extensively in recent years, which contains dynamic variables explicitly and does not commute with each other. Additional symmetries of the Kadomtsev-Petviashvili (KP) hierarchy was introduced by Orlov and Shulman [7] which contain one kind of important symmetry called Virasoro symmetry.

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These symmetries form a centerless $W_{1+\infty}$ algebra is closely related to matrix model by means of the Virasoro constraint and string equation[8, 9, 10, 11, 12]. Two sub-hierarchies of KP, BKP hierarchy and CKP hierarchy[13, 14], have been shown to possess additional symmetry[15]-[23] with consideration of the reductions on the Lax operators.

The 2-dimensional Toda Lattice(2dTL) hierarchy is introduced by Ueno and Takasaki in [24] based on the Sato theory. It is natural to construct the additional symmetry of the 2dTL hierarchy because of the similarity between the KP hierarchy and 2dTL hierarchy[12]. For the dispersionless Toda hierarchy[25, 26], additional symmetry is used to give string equations and Riemann-Hilbert problem. Note that, there exist two different sub-hierarchies of 2dTL hierarchy, 2-dimensional B type Toda Lattice (2dBTL) and 2-dimensional C type Toda Lattice (2dCTL) [24] which correspond to the infinite-dimensional algebras $w_\infty^B \times w_\infty^B$ and $w_\infty^C \times w_\infty^C$. The additional symmetry of the 2dBTL and 2dCTL hierarchies have been given recently in [27]. These results show that additional symmetry is one kind of general features of the integrable hierarchies.

As a generalization of Virasoro algebra, Block type infinite-dimensional Lie algebra and its representation theory have been studied intensively in references[28]-[35]. Note that the Virasoro algebra is one kind of widely used infinite-dimensional algebra in mathematical physics, particularly in integrable systems[36]. However, it is curious to note that, in the past 50 years after the introduction of the Block algebra, there does not exist a result on the application of this algebra in integrable systems until last year, to the best of our knowledge. In paper[37], we provide a novel Block type algebraic structure of the bigraded Toda hierarchy(BTH) with the help of the additional symmetry. This is the first time to find the direct relation between integrable hierarchy and the Block type algebra. Here BTH as a general reduction of 2dTL hierarchy, is introduced in [38, 39] from the background of the topological field and Gromov-Witten invariants. The Hirota bilinear equations and solutions of the BTH are given in [40, 41]. Later Block algebra is found again in dispersionless bigraded Toda hierarchy [42], in two-dimensional Toda hierarchy[43]. Very recently Block algebra has been shown to have a close relation with 3-algebra[44].

Based on the above results of Block algebra, in order to explore the universality of the Block type algebra in integrable systems, it is necessary to find this kind of algebraic structure in the KP type differential systems, due to the importance of the KP systems. In [45], one kind of additional symmetry of the KP hierarchy composed one generalized $W_{1+\infty}$ algebra with complicated structure coefficients but it was not a Block algebra. Taking into consideration the complexity of the relevant formula of the Block algebra, as well as plenty of different possible extensions of the KP hierarchy, it is a challenging problem to find this algebra in the KP type hierarchies.

It is a direct idea to consider the multi-component KP hierarchy(mcKP)[46, 47, 48, 49], which has a sole Lax operator with matrix coefficients. However, the algebra structure of the additional symmetry of the mcKP is very complicated and belongs to the Virasoro type[50]. Note that integrable systems possessing the symmetry of the Block type need to have two independent hierarchies of flows defined by two different Lax operators[37, 42, 43]. But the flows of the mcKP hierarchy with two pseudo-differential Lax operators are not well defined. Fortunately, a two-component BKP hierarchy with two Lax operators has been

constructed in [51] (see eq.(3.3) and eq.(3.9) of this reference) from the view of the Drinfeld-Sokolov Hierarchies of D Type. The Hamiltonian structure of this two-component BKP hierarchy is given in [52]. Therefore this two component BKP hierarchy [51] is a good candidate for us to explore the Block algebra in integrable hierarchy of KP type. In the following text of this paper we construct the generalized additional symmetries of the two-component BKP hierarchy and identify its algebraic structure by using a similar method in [43, 53]. Besides, the D type Drinfeld-Sokolov hierarchy is found to be a good differential model to derive complete Block type infinite dimensional Lie algebra.

This paper is arranged as follows. In next section we recall some necessary facts of the two-component BKP hierarchy. In Sections 3, we will give the generalized additional symmetries for the two-component BKP hierarchy. By reducing the two-component BKP hierarchy to the D type Drinfeld-Sokolov hierarchy, some concepts and results about this reduced hierarchy will be introduced in Section 4. The Block symmetries of Drinfeld-Sokolov hierarchy of type D will be derived in Section 5.

2. TWO COMPONENT BKP HIERARCHY

Let us firstly recall some basic facts [51, 54] of the two-component BKP hierarchy which is well defined by two Lax operators.

\mathcal{A} is assumed as an algebra of smooth functions of a spatial coordinate x and derivation denoted as $D = d/dx$. This algebra \mathcal{A} has following multiplying rule

$$D^i \cdot f = \sum_{r \geq 0} \binom{i}{r} D^r(f) D^{i-r}, \quad f \in \mathcal{A}.$$

For any operator $A = \sum_{i \in \mathbb{Z}} f_i D^i \in \mathcal{A}$, its nonnegative projection, negative projection, adjoint operator are respectively defined as

$$A_+ = \sum_{i \geq 0} f_i D^i, \quad A_- = \sum_{i < 0} f_i D^i, \quad A^* = \sum_{i \in \mathbb{Z}} (-D)^i \cdot f_i. \quad (2.1)$$

Basing on definition in [51], the two Lax operators of the two-component BKP hierarchy have form

$$L = D + \sum_{i \geq 1} u_i D^{-i}, \quad \hat{L} = D^{-1} \hat{u}_{-1} + \sum_{i \geq 1} \hat{u}_i D^i, \quad (2.2)$$

such that

$$L^* = -D L D^{-1} \quad \hat{L}^* = -D \hat{L} D^{-1}, \quad r \in \mathbb{Z}_+. \quad (2.3)$$

We call eq.(2.3) the B type condition of two-component BKP hierarchy.

The two-component BKP hierarchy is defined by the following Lax equations:

$$\frac{\partial \bar{L}}{\partial t_k} = [(L^k)_+, \bar{L}], \quad \frac{\partial \bar{L}}{\partial \hat{t}_k} = [-(\hat{L}^k)_-, \bar{L}] \quad (2.4)$$

with $\bar{L} = L$ or \hat{L} , $k \in \mathbb{Z}_+^{\text{odd}}$.

Note that $\partial/\partial t_1$ flow is equivalent to $\partial/\partial x$ flow, therefore it is reasonable to assume $t_1 = x$ in the following several sections.

One can write the operators L and \hat{L} in a dressing form as

$$L = \Phi D \Phi^{-1}, \quad \hat{L} = \hat{\Phi} D^{-1} \hat{\Phi}^{-1}, \quad (2.5)$$

where

$$\Phi = 1 + \sum_{i \geq 1} a_i D^{-i}, \quad \hat{\Phi} = 1 + \sum_{i \geq 1} b_i D^i \quad (2.6)$$

satisfy

$$\Phi^* = D\Phi^{-1}D^{-1}, \quad \hat{\Phi}^* = D\hat{\Phi}^{-1}D^{-1}. \quad (2.7)$$

Given L and \hat{L} , the dressing operators Φ and $\hat{\Phi}$ are determined uniquely up to a multiplication to the right by operators with constant coefficients. The two-component BKP hierarchy (2.4) can also be redefined as

$$\frac{\partial \Phi}{\partial t_k} = -(L^k)_- \Phi, \quad \frac{\partial \hat{\Phi}}{\partial t_k} = ((L^k)_+ - \delta_{k1} \hat{L}^{-1}) \hat{\Phi}, \quad (2.8)$$

$$\frac{\partial \Phi}{\partial \hat{t}_k} = -(\hat{L}^k)_- \Phi, \quad \frac{\partial \hat{\Phi}}{\partial \hat{t}_k} = (\hat{L}^k)_+ \hat{\Phi}, \quad (2.9)$$

with $k \in \mathbb{Z}_+^{\text{odd}}$.

Denote $\mathbf{t} = (t_1, t_3, t_5, \dots)$, $\hat{\mathbf{t}} = (\hat{t}_1, \hat{t}_3, \hat{t}_5, \dots)$ and introduce two wave functions

$$w(z) = w(\mathbf{t}, \hat{\mathbf{t}}; z) = \Phi e^{\xi(\mathbf{t}; z)}, \quad \hat{w}(z) = \hat{w}(\mathbf{t}, \hat{\mathbf{t}}; z) = \hat{\Phi} e^{xz + \xi(\hat{\mathbf{t}}; -z^{-1})}, \quad (2.10)$$

where the function ξ is defined as $\xi(\mathbf{t}; z) = \sum_{k \in \mathbb{Z}_+^{\text{odd}}} t_k z^k$. It is easy to see $D^i e^{xz} = z^i e^{xz}$, $i \in \mathbb{Z}$ and

$$L w(z) = z w(z), \quad \hat{L} \hat{w}(z) = z^{-1} \hat{w}(z).$$

The two-component BKP hierarchy was proved to have infinitely many bi-Hamiltonian structures and Hamiltonian densities which are the residues of L^k and \hat{L}^k with tau-symmetric condition[52]. The tau function of the two-component BKP hierarchy can be defined in form of the wave functions as

$$w(\mathbf{t}, \hat{\mathbf{t}}; z) = \frac{\tau(\mathbf{t} - 2[z^{-1}], \hat{\mathbf{t}})}{\tau(\mathbf{t}, \hat{\mathbf{t}})} e^{\xi(\mathbf{t}; z)}, \quad \hat{w}(\mathbf{t}, \hat{\mathbf{t}}; z) = \frac{\tau(\mathbf{t}, \hat{\mathbf{t}} + 2[z])}{\tau(\mathbf{t}, \hat{\mathbf{t}})} e^{\xi(\hat{\mathbf{t}}; -z^{-1})} \quad (2.11)$$

where $[z] = (z, z^3/3, z^5/5, \dots)$.

With above preparation, it is time to construct generalized additional symmetries for the two-component BKP hierarchy in the next section.

3. GENERALIZED ADDITIONAL SYMMETRIES OF THE TWO-COMPONENT BKP HIERARCHY

In this section, we are to construct generalized additional symmetries for the two-component BKP hierarchy by using the Orlov–Schulman operators whose coefficients depend explicitly on the time variables of the hierarchy.

With the same dressing operators given in eq.(2.6), Orlov–Schulman operators M, \hat{M} are constructed in following dressing structure [7, 54]

$$M = \Phi \Gamma \Phi^{-1}, \quad \hat{M} = \hat{\Phi} \hat{\Gamma} \hat{\Phi}^{-1},$$

where

$$\Gamma = \sum_{k \in \mathbb{Z}_+^{\text{odd}}} k t_k D^{k-1}, \quad \hat{\Gamma} = x + \sum_{k \in \mathbb{Z}_+^{\text{odd}}} k \hat{t}_k D^{-k-1}.$$

Then it is easy to get the following lemma.

Lemma 3.1. *The operators M and \hat{M} satisfy*

$$[L, M] = 1, \quad [\hat{L}^{-1}, \hat{M}] = 1; \quad Mw(z) = \partial_z w(z), \quad \hat{M}\hat{w}(z) = \partial_z \hat{w}(z); \quad (3.1)$$

and for $\bar{M} = M$ or \hat{M} ,

$$\frac{\partial \bar{M}}{\partial t_k} = [(L^k)_+, \bar{M}], \quad \frac{\partial \bar{M}}{\partial \hat{t}_k} = [-(\hat{L}^k)_-, \bar{M}], \quad k \in \mathbb{Z}_+^{\text{odd}}. \quad (3.2)$$

For an operator $A = A(L, \hat{L}^{-1}, M, \hat{M})$ which can be written as an antisymmetric form as $A = B - D^{-1}B^*D$, define a flow Y_A acting on Φ and $\hat{\Phi}$ as

$$Y_A \Phi = -A_- \Phi, \quad Y_A \hat{\Phi} = A_+ \hat{\Phi}, \quad (3.3)$$

therefore

$$Y_A L = [-A_-, L], \quad Y_A \hat{L} = [A_+, \hat{L}],$$

and

$$Y_A M = [-A_-, M], \quad Y_A \hat{M} = [A_+, \hat{M}].$$

Following calculation can be easily got

$$[Y_A, Y_B] \Phi = -(Y_A B)_- \Phi + (Y_B A)_- \Phi + [B_-, A_-] \Phi,$$

$$[Y_A, Y_B] \hat{\Phi} = (Y_A B)_+ \hat{\Phi} - (Y_B A)_+ \hat{\Phi} + [B_+, A_+] \hat{\Phi}.$$

Above two identities can be written as an universal form

$$[Y_A, Y_B] \Phi(\hat{\Phi}) = Y_{\{B, A\}} \Phi(\hat{\Phi}), \quad (3.4)$$

where

$$\{A, B\} = -Y_A B + Y_B A - [A_-, B_-] + [A_+, B_+]. \quad (3.5)$$

Also one can derive following proposition.

Proposition 3.2. *For any polynomial $A = A(L, \hat{L}^{-1}, M, \hat{M})$, one has*

$$\frac{\partial A}{\partial t_k} = [(L^k)_+, A], \quad \frac{\partial A}{\partial \hat{t}_k} = [-(\hat{L}^k)_-, A], \quad k \in \mathbb{Z}_+^{\text{odd}}. \quad (3.6)$$

Proof Proof is easy to finish by considering eqs.(2.4) and eqs.(3.2). \square

Using eq.(3.3) and Proposition 3.2, it can be proved that the flow eqs.(3.3) can commute with original flow of the two-component BKP hierarchy, i.e.

$$[Y_A, \frac{\partial}{\partial t_k}] = 0, \quad [Y_A, \frac{\partial}{\partial \hat{t}_k}] = 0. \quad (3.7)$$

That means they are symmetries of the two-component BKP hierarchy. This kind of symmetries contain original additional $w_\infty^B \times w_\infty^B$ symmetry of the two-component BKP hierarchy mentioned in [54]. The definition of operator A here is more general than operators used to construct additional symmetry of two component BKP hierarchy in [54] because the multiplication mixed set $\{L, M\}$ and set $\{\hat{L}, \hat{M}\}$ together. Therefore we call it the generalized additional symmetry of the two-component BKP hierarchy. Here we will not give a detailed

proof of this symmetry but later we will prove some special symmetry of this kind of generalized additional symmetries.

The new bracket structure $\{ , \}$ can be expressed by the standard bracket structure $[,]$ which is showed in the following lemma.

Lemma 3.3. *Following relations between two bracket structure hold*

$$\{f\hat{f}, g\hat{g}\} = [f, g]\hat{f}\hat{g} - fg[\hat{f}, \hat{g}], \quad (3.8)$$

$$\{\hat{f}\hat{f}, \hat{g}\hat{g}\} = \hat{f}\hat{g}[f, g] - [\hat{f}, \hat{g}]fg, \quad (3.9)$$

$$\{f\hat{f}, \hat{g}\hat{g}\} = [f, \hat{g}][g, \hat{f}] - f[\hat{f}, \hat{g}]g + \hat{g}[f, g]\hat{f}, \quad (3.10)$$

where f, g are polynomials of L, M and \hat{f}, \hat{g} are polynomials of \hat{L}, \hat{M} .

Proof The first two identities can be easily derived by direct calculation basing on definition, therefore we only give the proof of the identity (3.10) as following

$$\begin{aligned} \{f\hat{f}, \hat{g}\hat{g}\} &= -Y_{f\hat{f}}(\hat{g}\hat{g}) + Y_{\hat{g}\hat{g}}(f\hat{f}) + [(f\hat{f})_+, (\hat{g}\hat{g})_+] - [(f\hat{f})_-, (\hat{g}\hat{g})_-] \\ &= -[(f\hat{f})_+, \hat{g}]g + \hat{g}[(f\hat{f})_-, g] - [(\hat{g}\hat{g})_-, \hat{f}]\hat{f} + f[(\hat{g}\hat{g})_+, \hat{f}] \\ &\quad + [(f\hat{f})_+, (\hat{g}\hat{g})_+] - [(f\hat{f})_-, (\hat{g}\hat{g})_-] \\ &= f\hat{g}g\hat{f} + \hat{g}f\hat{f}g - f\hat{f}\hat{g}g - \hat{g}g\hat{f}\hat{f} \\ &= (f\hat{g} - \hat{g}f)(g\hat{f} - \hat{f}g) - f\hat{f}\hat{g}g + f\hat{g}\hat{f}g + \hat{g}f\hat{f}g - \hat{g}g\hat{f}\hat{f} \\ &= [f, \hat{g}][g, \hat{f}] - f[\hat{f}, \hat{g}]g + \hat{g}[f, g]\hat{f}. \end{aligned}$$

□

From eq.(3.4), it is easy to see following lemma holds.

Lemma 3.4. *There is an antihomomorphism between two sets, i.e. $\mathbb{C}[L, \hat{L}, M, \hat{M}]$ and $\mathcal{G} = \{Y_A | A = A(L, \hat{L}^{-1}, M, \hat{M})\}$*

$$\begin{aligned} \mathbb{C}[L, \hat{L}, M, \hat{M}], \{ \} &\mapsto \mathcal{G}, [], \\ A &\mapsto Y_A, \end{aligned}$$

which satisfy following antihomomorphism relation

$$[Y_A, Y_B]\Phi(\hat{\Phi}) = Y_{\{B, A\}}\Phi(\hat{\Phi}).$$

Because of the anti-order of spectral representation of multiplications of Lax operators and Orlov-Schulman operators, following lemmas can be easily derived.

Lemma 3.5. *For $a_1, a_2, b_1, b_2 \in \mathbb{Z}_+$, there is an anti homomorphism*

$$\begin{aligned} \omega_\infty \otimes \omega_\infty, [] &\mapsto \mathbb{C}[L, \hat{L}^-, M, \hat{M}], \{ \}, \\ z_1^{a_1} \partial_{z_1}^{b_1} z_2^{a_2} \partial_{z_2}^{b_2} &\mapsto M^{b_1} L^{a_1} \hat{L}^{-a_2} \hat{M}^{b_2}, \\ [z_1^{a_1} \partial_{z_1}^{b_1} z_2^{a_2} \partial_{z_2}^{b_2}, z_1^{c_1} \partial_{z_1}^{d_1} z_2^{c_2} \partial_{z_2}^{d_2}] &\mapsto \{M^{d_1} L^{c_1} \hat{L}^{-d_2} \hat{M}^{d_2}, M^{b_1} L^{a_1} \hat{L}^{-a_2} \hat{M}^{b_2}\}. \end{aligned}$$

Lemma 3.6. *For $a_1, a_2, b_1, b_2 \in \mathbb{Z}_+$, there is an isomorphism*

$$\begin{aligned} \psi : \omega_\infty \otimes \omega_\infty, [] &\mapsto \mathcal{G}, [], \\ z_1^{a_1} \partial_{z_1}^{b_1} z_2^{a_2} \partial_{z_2}^{b_2} &\mapsto Y_{M^{b_1} L^{a_1} \hat{L}^{-a_2} \hat{M}^{b_2}}, \\ [z_1^{a_1} \partial_{z_1}^{b_1} z_2^{a_2} \partial_{z_2}^{b_2}, z_1^{c_1} \partial_{z_1}^{d_1} z_2^{c_2} \partial_{z_2}^{d_2}] &\mapsto Y_{\{M^{d_1} L^{c_1} \hat{L}^{-d_2} \hat{M}^{d_2}, M^{b_1} L^{a_1} \hat{L}^{-a_2} \hat{M}^{b_2}\}}, \end{aligned}$$

i.e.

$$[\psi(z_1^{a_1} \partial_{z_1}^{b_1} z_2^{a_2} \partial_{z_2}^{b_2}), \psi(z_1^{c_1} \partial_{z_1}^{c_1} z_2^{d_2} \partial_{z_2}^{d_2})] = \psi([z_1^{a_1} \partial_{z_1}^{b_1} z_2^{a_2} \partial_{z_2}^{b_2}, z_1^{c_1} \partial_{z_1}^{d_1} z_2^{d_2} \partial_{z_2}^{d_2}]). \quad (3.11)$$

From now on, we will introduce one special kind of case of $A = A(L, \hat{L}^{-1}, M, \hat{M})$, i.e. the following two operators B_{ml} and \hat{B}_{ml} . Given any pair of integers (m, l) with $m, l \geq 0$, define

$$B_{ml} = ML^{m+1} \hat{L}^{-l} + (-1)^{l+m} \hat{L}^{-l} L^m ML, \quad (3.12)$$

$$\hat{B}_{ml} = L^m \hat{L}^{-l+1} \hat{M} + (-1)^{l+m} \hat{L} \hat{M} \hat{L}^{-l} L^m. \quad (3.13)$$

The definitions of B_{ml} and \hat{B}_{ml} are also different from definitions in [54]. As a corollary of Proposition 3.2, following proposition can be got.

Proposition 3.7. *For any $\bar{B}_{ml} = B_{ml}, \hat{B}_{ml}$, one has*

$$\frac{\partial \bar{B}_{ml}}{\partial t_k} = [(L^k)_+, \hat{B}_{ml}], \quad \frac{\partial \bar{B}_{ml}}{\partial \hat{t}_k} = [-(\hat{L}^k)_-, B_{ml}], \quad k \in \mathbb{Z}_+^{\text{odd}}. \quad (3.14)$$

To prove that B_{ml} and \hat{B}_{ml} satisfy B type condition, we need following lemma.

Lemma 3.8. *Operators M and \hat{M} satisfy following conjugate identities,*

$$M^* = DL^{-1}MLD^{-1}, \quad \hat{M}^* = D\hat{L}\hat{M}\hat{L}^{-1}D^{-1}. \quad (3.15)$$

Proof Using

$$\Phi^* = D\Phi^{-1}D^{-1}, \quad \hat{\Phi}^* = D\hat{\Phi}^{-1}D^{-1},$$

following calculations

$$M^* = \Phi^{*-1}\Gamma\Phi^* = D\Phi D^{-1}\Gamma D\Phi^{-1}D^{-1} = D\Phi D^{-1}\Phi^{-1}M\Phi D\Phi^{-1}D^{-1},$$

$$\hat{M}^* = \hat{\Phi}^{*-1}\hat{\Gamma}\hat{\Phi}^* = D\hat{\Phi} D^{-1}\hat{\Gamma} D\hat{\Phi}^{-1}D^{-1} = D\hat{\Phi} D^{-1}\hat{\Phi}^{-1}\hat{M}\hat{\Phi} D\hat{\Phi}^{-1}D^{-1},$$

will lead to this lemma. \square

It is easy to check following proposition holds basing on the Lemma 3.8 above.

Proposition 3.9. *B_{ml} and \hat{B}_{ml} satisfy B type condition, namely*

$$B_{ml}^* = -DB_{ml}D^{-1}, \quad \hat{B}_{ml}^* = -D\hat{B}_{ml}D^{-1}. \quad (3.16)$$

Proof Using Proposition 3.8, following calculation will lead to first identity of this proposition

$$\begin{aligned} B_{ml}^* &= (ML^{m+1} \hat{L}^{-l} - (-1)^{l+m+1} \hat{L}^{-l} L^m ML)^* \\ &= \hat{L}^{-l*} L^{m+1*} M^* - (-1)^{l+m+1} L^{1*} M^* L^{m*} \hat{L}^{-l*} \\ &= (-1)^{l+m+1} D\hat{L}^{-l} L^m MLD^{-1} - DML^{m+1} \hat{L}^{-l} D^{-1} \\ &= -D(ML^{m+1} \hat{L}^{-l} - (-1)^{l+m+1} \hat{L}^{-l} L^m ML)D^{-1}. \end{aligned}$$

The second identity can be proved in similar way. \square

Because of Proposition 3.9, the following equations are well defined

$$\frac{\partial \Phi}{\partial b_{ml}} = Y_{B_{ml}} \Phi = -(B_{ml})_- \Phi, \quad \frac{\partial \hat{\Phi}}{\partial \hat{b}_{ml}} = Y_{\hat{B}_{ml}} \hat{\Phi} = (\hat{B}_{ml})_+ \hat{\Phi}, \quad (3.17)$$

$$\frac{\partial \Phi}{\partial \hat{b}_{ml}} = Y_{\hat{B}_{ml}} \Phi = -(\hat{B}_{ml})_- \Phi, \quad \frac{\partial \hat{\Phi}}{\partial b_{ml}} = Y_{B_{ml}} \hat{\Phi} = (B_{ml})_+ \hat{\Phi}. \quad (3.18)$$

These equations are equivalent to following Lax equations

$$\frac{\partial L}{\partial b_{ml}} = [-(B_{ml})_-, L], \quad \frac{\partial \hat{L}}{\partial b_{ml}} = [(B_{ml})_+, \hat{L}], \quad (3.19)$$

$$\frac{\partial L}{\partial \hat{b}_{ml}} = [-(\hat{B}_{ml})_-, L], \quad \frac{\partial \hat{L}}{\partial \hat{b}_{ml}} = [(\hat{B}_{ml})_+, \hat{L}]. \quad (3.20)$$

These flows are in fact some special cases of generalized additional symmetries eqs.(3.3). To show some techniques in the proof of generalized symmetry eq.(3.7), we will give a short proof of following proposition.

Proposition 3.10. *The flows (3.19) and (3.20) commute with the flows of the two-component BKP hierarchy. Namely, for any $\bar{b}_{ml} = b_{ml}, \hat{b}_{ml}$ and $\bar{t}_k = t_k, \hat{t}_k$ one has*

$$\left[\frac{\partial}{\partial \bar{b}_{ml}}, \frac{\partial}{\partial \bar{t}_k} \right] = 0, \quad m, l \in \mathbb{Z}_+, \quad k \in \mathbb{Z}_+^{\text{odd}}. \quad (3.21)$$

Proof The proposition can be checked case by case with the help of eq.(3.14) and eqs.(3.19)-(3.20). For example,

$$\begin{aligned} & \left[\frac{\partial}{\partial \bar{b}_{ml}}, \frac{\partial}{\partial \bar{t}_k} \right] \Phi \\ &= [(\hat{L}^k)_-, (B_{ml})_-] \Phi - [(B_{ml})_+, \hat{L}^k]_- \Phi - [(\hat{L}^k)_-, B_{ml}]_- \Phi = 0, \\ & \left[\frac{\partial}{\partial \hat{b}_{ml}}, \frac{\partial}{\partial \hat{t}_k} \right] \hat{\Phi} \\ &= [(L^k)_+ - \delta_{k1} \hat{L}^{-1}, (\hat{B}_{ml})_+] \hat{\Phi} + \left([-(\hat{B}_{ml})_-, L^k]_+ - \delta_{k1} [(\hat{B}_{ml})_+, \hat{L}^{-1}] \right) \hat{\Phi} \\ & \quad - [(L^k)_+, \hat{B}_{ml}]_+ \hat{\Phi} = 0. \end{aligned}$$

The other cases can be proved in similar ways. This is the end of this proposition. \square

This proposition implies that the additional flows (3.19)-(3.20) are symmetries of the two-component BKP hierarchy. To see the further structure of the additional symmetry, we need following proposition.

Proposition 3.11. *The operators $\bar{B}_{m,n} = B_{m,n}, \hat{B}_{m,n}, m, n \in \mathbb{Z}_+$ of the two-component BKP hierarchy satisfy following identity*

$$\begin{aligned} \{B_{m_1, m_2}, B_{n_1, n_2}\} &= (m_1 - n_1) B_{m_1+n_1, m_2+n_2} + Q_{m_1, m_2, n_1, n_2}, \\ \{\hat{B}_{m_1, m_2}, \hat{B}_{n_1, n_2}\} &= (m_2 - n_2) \hat{B}_{m_1+n_1, m_2+n_2} + \hat{Q}_{m_1, m_2, n_1, n_2}, \\ \{B_{m_1, m_2}, \hat{B}_{n_1, n_2}\} &= -n_1 \hat{B}_{m_1+n_1, m_2+n_2} + m_2 B_{m_1+n_1, m_2+n_2} + \bar{Q}_{m_1, m_2, n_1, n_2}, \end{aligned}$$

where

$$\begin{aligned} Q_{m_1, m_2, n_1, n_2} &= (-1)^{m_2+m_1+1} \{ML^{n_1+1} \hat{L}^{-n_2}, \hat{L}^{-m_2} L^{m_1} ML\} \\ & \quad + (-1)^{n_2+n_1} \{ML^{m_1+1} \hat{L}^{-m_2}, \hat{L}^{-n_2} L^{n_1} ML\}, \end{aligned}$$

$$\hat{Q}_{m_1, m_2, n_1, n_2} = (-1)^{m_2+m_1+1} \{L^{n_1} \hat{L}^{-n_2+1} \hat{M}, \hat{L} \hat{M} \hat{L}^{-m_2} L^{m_1}\}$$

$$+(-1)^{n_2+n_1}\{L^{m_1}\hat{L}^{-m_2+1}\hat{M}, \hat{L}\hat{M}\hat{L}^{-n_2}L^{n_1}\},$$

$$\begin{aligned}\bar{Q}_{m_1,m_2,n_1,n_2} = & (-1)^{m_2+m_1+1}\{L^{n_1}\hat{L}^{-n_2+1}\hat{M}, \hat{L}^{-m_2}L^{m_1}ML\} \\ & +(-1)^{n_2+n_1}\{ML^{m_1+1}\hat{L}^{-m_2}, \hat{L}\hat{M}\hat{L}^{-n_2}L^{n_1}\}.\end{aligned}$$

Take

$$\tilde{B}_{m_1,m_2} = (m_2 + 1)B_{m_1,m_2} - m_1\hat{B}_{m_1,m_2},$$

then

$$[\tilde{B}_{m_1,m_2}, \tilde{B}_{n_1,n_2}] = ((n_2 + 1)m_1 - (m_2 + 1)n_1)\tilde{B}_{m_1+n_1,m_2+n_2} + S_{m_1,m_2,n_1,n_2},$$

where

$$\begin{aligned}S_{m_1,m_2,n_1,n_2} = & (m_2 + 1)(n_2 + 1)Q_{m_1,m_2,n_1,n_2} - n_1(m_2 + 1)\bar{Q}_{m_1,m_2,n_1,n_2} \\ & + m_1(n_2 + 1)\bar{Q}_{n_1,n_2,m_1,m_2} + m_1n_1\hat{Q}_{m_1,m_2,n_1,n_2}.\end{aligned}$$

Then the following theorem is clear.

Theorem 3.12. *The additional flows (3.19) and (3.20) satisfy*

$$[\partial_{b_{m_1,m_2}}, \partial_{b_{n_1,n_2}}] = (m_1 - n_1)\partial_{b_{m_1+n_1,m_2+n_2}} + Y_{Q_{m_1,m_2,n_1,n_2}}, \quad (3.22)$$

$$[\partial_{\hat{b}_{m_1,m_2}}, \partial_{\hat{b}_{n_1,n_2}}] = (m_2 - n_2)\partial_{\hat{b}_{m_1+n_1,m_2+n_2}} + Y_{\hat{Q}_{m_1,m_2,n_1,n_2}}, \quad (3.23)$$

$$[\partial_{b_{m_1,m_2}}, \partial_{\hat{b}_{n_1,n_2}}] = m_2\partial_{b_{m_1+n_1,m_2+n_2}} - n_1\partial_{\hat{b}_{m_1+n_1,m_2+n_2}} + Y_{\bar{Q}_{m_1,m_2,n_1,n_2}}, \quad (3.24)$$

$$[\partial_{v_{m_1,m_2}}, \partial_{v_{n_1,n_2}}] = ((n_2 + 1)m_1 - (m_2 + 1)n_1)\partial_{v_{m_1+n_1,m_2+n_2}} + Y_{S_{m_1,m_2,n_1,n_2}}, \quad (3.25)$$

where

$$\partial_{v_{m_1,m_2}} = (m_2 + 1)\partial_{b_{m_1,m_2}} - m_1\partial_{\hat{b}_{n_1,n_2}}.$$

Without B type condition eq.(2.3), the operators $(Q, \hat{Q}, S, Y_Q, Y_{\hat{Q}}, Y_S)$ will vanish. This will lead to nice Block symmetric structure. Proposition 3.10 and Theorem 3.12 show that the flows (3.19) and (3.20) give one modified Block type additional symmetries for the two-component BKP hierarchy. The obstacle to derive the perfect Block type symmetry is due to the constrained B type condition eq.(2.3) of the two-component BKP hierarchy. That means if we only consider the the two-component KP hierarchy, i.e. the two-component BKP hierarchy without the constrained B type condition, the additional symmetry will compose nice structure of Block type infinite dimensional Lie algebra. But unfortunately the Lax representation with two different pseudo-differential operators of the two-component KP hierarchy is not well-defined.

If we choose $l = 0$ in the operator $B_{m,l}$, $m = 0$ in the operator $\hat{B}_{m,l}$ and increase one index on each operator of M, \hat{M} , then this symmetry will be the $w_\infty^B \times w_\infty^B$ algebra mentioned in [54].

Although we only get modified Block type additional symmetries for the two-component BKP hierarchy, further calculation in the next section supports: If we do a $(2n,2)$ -reduction from the two-component BKP hierarchy, perfect Block type additional symmetry will be exactly kept. This reduced hierarchy is nothing but the D type Drinfeld–Sokolov hierarchies [55] which will be discussed in the next section.

4. D TYPE DRINFELD–SOKOLOV HIERARCHY

Assume a new Lax operator \mathcal{L} which has following relation with two Lax operators of the two-component BKP hierarchy introduced in last section

$$\mathcal{L} = L^{2n} = \hat{L}^2. \quad (4.1)$$

Then the Lax operators of two-component BKP hierarchy will be reduced to the following Lax operator of D type Drinfeld–Sokolov hierarchy[51]

$$\mathcal{L} = D^{2n} + \frac{1}{2} \sum_{i=1}^n D^{-1} (v_i D^{2i-1} + D^{2i-1} v_i) + D^{-1} \rho D^{-1} \rho. \quad (4.2)$$

One can easily find the Lax operator \mathcal{L} of D type Drinfeld–Sokolov hierarchy will not satisfy the reduction condition as Lax operator of the two-component BKP hierarchy but satisfy following B type condition

$$\mathcal{L}^* = D \mathcal{L} D^{-1}. \quad (4.3)$$

This Lax operator \mathcal{L} of D type Drinfeld–Sokolov hierarchy has following dressing structure[54]

$$\mathcal{L} = \Phi D^{2n} \Phi^{-1} = \hat{\Phi} D^{-2} \hat{\Phi}^{-1}. \quad (4.4)$$

The dressing structure inspire us to define two fractional operators as

$$\mathcal{L}^{\frac{1}{2n}} = D + \sum_{i \geq 1} u_i D^{-i}, \quad \mathcal{L}^{\frac{1}{2}} = D^{-1} \hat{u}_{-1} + \sum_{i \geq 1} \hat{u}_i D^i. \quad (4.5)$$

The D type Drinfeld–Sokolov hierarchy being considered in this paper is defined by the following Lax equations:

$$\frac{\partial \mathcal{L}}{\partial t_k} = [(\mathcal{L}^{\frac{k}{2n}})_+, \mathcal{L}], \quad \frac{\partial \mathcal{L}}{\partial \hat{t}_k} = [-(\mathcal{L}^{\frac{k}{2}})_-, \mathcal{L}], \quad k \in \mathbb{Z}_+^{\text{odd}}. \quad (4.6)$$

Among these hierarchies, the Drinfeld–Sokolov hierarchy of type D_n is associated to the affine algebra $D_n^{(1)}$ and the zeroth vertex of its Dynkin diagram [55, 51]. Similarly as the two-component BKP hierarchy, the equivalence between $\partial/\partial t_1$ and $\partial/\partial x$ leads to assumption as $t_1 = x$.

Two fractional operators $\mathcal{L}^{\frac{1}{2n}}$ and $\mathcal{L}^{\frac{1}{2}}$ can be rewritten in a dressing form as

$$\mathcal{L}^{\frac{1}{2n}} = \Phi D \Phi^{-1}, \quad \mathcal{L}^{\frac{1}{2}} = \hat{\Phi} D^{-1} \hat{\Phi}^{-1}. \quad (4.7)$$

Here

$$\Phi = 1 + \sum_{i \geq 1} a_i D^{-i}, \quad \hat{\Phi} = 1 + \sum_{i \geq 1} b_i D^i \quad (4.8)$$

are pseudo-differential operators that also satisfy following B type condition

$$\Phi^* = D \Phi^{-1} D^{-1}, \quad \hat{\Phi}^* = D \hat{\Phi}^{-1} D^{-1}. \quad (4.9)$$

The dressing operators Φ and $\hat{\Phi}$ are same as the ones of two-component BKP hierarchy. Given \mathcal{L} , the dressing operators Φ and $\hat{\Phi}$ are uniquely determined up to a multiplication to the right by operators of the form (4.8) and (4.9) with constant coefficients. The D type Drinfeld–Sokolov hierarchies can also be redefined as

$$\frac{\partial \Phi}{\partial t_k} = -(\mathcal{L}^{\frac{k}{2n}})_- \Phi, \quad \frac{\partial \hat{\Phi}}{\partial \hat{t}_k} = ((\mathcal{L}^{\frac{k}{2n}})_+ - \delta_{k1} \mathcal{L}^{-\frac{1}{2}}) \hat{\Phi}, \quad (4.10)$$

$$\frac{\partial \Phi}{\partial \hat{t}_k} = -(\mathcal{L}^{\frac{k}{2}})_- \Phi, \quad \frac{\partial \hat{\Phi}}{\partial \hat{t}_k} = (\mathcal{L}^{\frac{k}{2}})_+ \hat{\Phi} \quad (4.11)$$

with $k \in \mathbb{Z}_+^{\text{odd}}$.

Denote $\mathbf{t} = (t_1, t_3, t_5, \dots)$ and $\hat{\mathbf{t}} = (\hat{t}_1, \hat{t}_3, \hat{t}_5, \dots)$ and introduce two wave functions

$$w(z^{\frac{1}{2n}}) = w(\mathbf{t}, \hat{\mathbf{t}}; z^{\frac{1}{2n}}) = \Phi e^{\xi(\mathbf{t}; z^{\frac{1}{2n}})}, \quad (4.12)$$

$$\hat{w}(z^{\frac{1}{2}}) = \hat{w}(\mathbf{t}, \hat{\mathbf{t}}; z^{\frac{1}{2}}) = \hat{\Phi} e^{xz^{\frac{1}{2}} + \xi(\hat{\mathbf{t}}; -z^{-\frac{1}{2}})}, \quad (4.13)$$

where the function ξ is defined as $\xi(\mathbf{t}; z) = \sum_{k \in \mathbb{Z}_+^{\text{odd}}} t_k z^k$, and $D^i e^{xz} = z^i e^{xz}$ for any $i \in \mathbb{Z}$. It is easy to see

$$\mathcal{L} w(z^{\frac{1}{2n}}) = z w(z^{\frac{1}{2n}}), \quad \mathcal{L} \hat{w}(z^{\frac{1}{2}}) = z^{-1} \hat{w}(z^{\frac{1}{2}}).$$

After above preparation, we will show that this D type Drinfeld-Sokolov hierarchies have nice Block symmetry as its appearance in BTH [37].

5. BLOCK SYMMETRIES OF D TYPE DRINFELD-SOKOLOV HIERARCHIES

In this section, we will put constrained condition eq.(4.1) into construction of the flows of additional symmetry which form the well-known Block algebra.

With the dressing operators given in eq.(4.7), we introduce Orlov-Schulman operators as following

$$\mathcal{M} = \Phi \Gamma_L \Phi^{-1}, \quad \hat{\mathcal{M}} = \hat{\Phi} \hat{\Gamma}_R \hat{\Phi}^{-1},$$

where

$$\Gamma_L = \sum_{k \in \mathbb{Z}_+^{\text{odd}}} \frac{k}{2n} t_k D^{k-2n}, \quad \hat{\Gamma}_R = -\frac{x}{2} D^3 - \frac{1}{2} \sum_{k \in \mathbb{Z}_+^{\text{odd}}} k \hat{t}_k D^{2-k}.$$

It is easy to see the following lemma holds.

Lemma 5.1. *The operators \mathcal{M} and $\hat{\mathcal{M}}$ satisfy*

$$[\mathcal{L}, \mathcal{M}] = 1, \quad [\mathcal{L}, \hat{\mathcal{M}}] = 1; \quad (5.1)$$

and

$$\mathcal{M} w(z^{\frac{1}{2n}}) = \partial_z w(z^{\frac{1}{2n}}), \quad \hat{\mathcal{M}} \hat{w}(z^{\frac{1}{2}}) = \partial_{z^{-1}} \hat{w}(z^{\frac{1}{2}}); \quad (5.2)$$

$$\frac{\partial \bar{\mathcal{M}}}{\partial t_k} = [(\mathcal{L}^{\frac{k}{2n}})_+, \bar{\mathcal{M}}], \quad \frac{\partial \bar{\mathcal{M}}}{\partial \hat{t}_k} = [-(\mathcal{L}^{\frac{k}{2}})_-, \bar{\mathcal{M}}], \quad (5.3)$$

where $\bar{\mathcal{M}} = \mathcal{M}$ or $\hat{\mathcal{M}}$, $k \in \mathbb{Z}_+^{\text{odd}}$.

To make the operators used in additional symmetry satisfying B type condition, we need to prove the following B type property of $\mathcal{M} - \hat{\mathcal{M}}$ which is included in following lemma.

Lemma 5.2. *The difference of two Orlov-Schulman operators \mathcal{M} and $\hat{\mathcal{M}}$ for D type Drinfeld-Sokolov hierarchy has following D type property:*

$$\mathcal{L}^*(\mathcal{M} - \hat{\mathcal{M}})^* = -D\mathcal{L}(\mathcal{M} - \hat{\mathcal{M}})D^{-1}. \quad (5.4)$$

Proof It is easy to find the two Orlov-Schulman operators \mathcal{M} and $\hat{\mathcal{M}}$ of the D type Drinfeld-Sokolov hierarchy can be expressed by Orlov-Schulman operators M, \hat{M} and Lax operators L, \hat{L} of two-component BKP hierarchy as

$$\mathcal{M} = \frac{ML^{1-2n}}{2n}, \quad \hat{\mathcal{M}} = -\frac{\hat{M}\hat{L}^{-3}}{2}. \quad (5.5)$$

Using Lemma 3.8, putting eq.(5.5) into $(\mathcal{M} - \hat{\mathcal{M}})^*$ can lead to

$$(\mathcal{M} - \hat{\mathcal{M}})^* = -\frac{DL^{-2n}MLD^{-1}}{2n} - \frac{D\hat{L}^{-2}\hat{M}\hat{L}^{-1}D^{-1}}{2} \quad (5.6)$$

$$= -\frac{DL^{-2n}MLD^{-1}}{2n} - \frac{D\hat{L}^{-2}\hat{M}\hat{L}^{-1}D^{-1}}{2} \quad (5.7)$$

$$= -\frac{D(ML^{1-2n} - 2nL^{-2n})D^{-1}}{2n} - \frac{D(\hat{M}\hat{L}^{-3} + 2\hat{L}^{-2})D^{-1}}{2}, \quad (5.8)$$

which can further lead to

$$\mathcal{L}^*(\mathcal{M} - \hat{\mathcal{M}})^* = -D(\mathcal{L}\mathcal{M} - \mathcal{L}\hat{\mathcal{M}})D^{-1}. \quad (5.9)$$

In above calculation, the commutativity between \mathcal{L} and $\mathcal{M} - \hat{\mathcal{M}}$ is already used. Till now, the proof is finished. \square

For D-type Drinfeld-Sokolov hierarchy, m is supposed to be odd number to avoid being trivial and simplify it to

$$\mathcal{B}_{m,l} = (\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l, \quad m \in \mathbb{Z}_+^{\text{odd}}. \quad (5.10)$$

One can easily check that

$$\mathcal{B}_{m,l}^* = -D\mathcal{B}_{m,l}D^{-1}, \quad m \in \mathbb{Z}_+^{\text{odd}}. \quad (5.11)$$

That means it is reasonable to define additional flow of the D type Drinfeld-Sokolov hierarchy

$$\frac{\partial \mathcal{L}}{\partial c_{m,l}} = [-(\mathcal{B}_{m,l})_-, \mathcal{L}], \quad m \in \mathbb{Z}_+^{\text{odd}}, l \in \mathbb{Z}_+. \quad (5.12)$$

Proposition 5.3. *For the Drinfeld-Sokolov hierarchy of type D, the following statements hold true.*

(1) *The flows (5.12) can commute with original flow of the Drinfeld-Sokolov hierarchy of type D, namely,*

$$\left[\frac{\partial}{\partial c_{m,l}}, \frac{\partial}{\partial t_k} \right] = 0, \quad \left[\frac{\partial}{\partial c_{m,l}}, \frac{\partial}{\partial \hat{t}_k} \right] = 0, \quad l \in \mathbb{Z}_+, m, k \in \mathbb{Z}_+^{\text{odd}}.$$

(2)

$$\frac{\partial \mathcal{M}}{\partial c_{m,l}} = [-(\mathcal{B}_{m,l})_-, \mathcal{M}], \quad \frac{\partial \hat{\mathcal{M}}}{\partial c_{m,l}} = [(\mathcal{B}_{m,l})_+, \hat{\mathcal{M}}], \quad m \in \mathbb{Z}_+^{\text{odd}}, l \in \mathbb{Z}_+. \quad (5.13)$$

(3) *Let $w(z^{\frac{1}{2n}})$ and $\hat{w}(z^{\frac{1}{2}})$ be the wave functions defined as in (4.12), then*

$$\frac{\partial w(z^{\frac{1}{2n}})}{\partial c_{m,l}} = -(\mathcal{B}_{m,l})_- w(z^{\frac{1}{2n}}), \quad \frac{\partial \hat{w}(z^{\frac{1}{2}})}{\partial c_{m,l}} = (\mathcal{B}_{m,l})_+ \hat{w}(z^{\frac{1}{2}}), \quad j \geq -1. \quad (5.14)$$

Proof According to the definition,

$$[\partial_{c_{m,l}}, \partial_{t_k}] \Phi = \partial_{c_{m,l}}(\partial_{t_k} \Phi) - \partial_{t_k}(\partial_{c_{m,l}} \Phi),$$

and using the actions of the additional flows and the flows of D type Drinfeld-Sokolov hierarchy on Φ , we have

$$\begin{aligned} [\partial_{c_{m,l}}, \partial_{t_k}] \Phi &= -\partial_{c_{m,l}} \left((\mathcal{L}^{\frac{k}{2n}})_- \Phi \right) + \partial_{t_k} \left(((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_- \Phi \right) \\ &= -(\partial_{c_{m,l}} \mathcal{L}^{\frac{k}{2n}})_- \Phi - (\mathcal{L}^{\frac{k}{2n}})_- (\partial_{c_{m,l}} \Phi) \\ &\quad + [\partial_{t_k} ((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)]_- \Phi + ((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_- (\partial_{t_k} \Phi). \end{aligned}$$

Using eq.(4.6) and eq.(5.3), it equals

$$\begin{aligned} [\partial_{c_{m,l}}, \partial_{t_k}] \Phi &= [((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_-, \mathcal{L}^{\frac{k}{2n}}]_- \Phi + (\mathcal{L}^{\frac{k}{2n}})_- ((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_- \Phi \\ &\quad + [(\mathcal{L}^{\frac{k}{2n}})_+, (\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l]_- \Phi - ((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_- (\mathcal{L}^{\frac{k}{2n}})_- \Phi \\ &= [((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_-, \mathcal{L}^{\frac{k}{2n}}]_- \Phi - [(\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l, (\mathcal{L}^{\frac{k}{2n}})_+]_- \Phi \\ &\quad + [(\mathcal{L}^{\frac{k}{2n}})_-, ((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_-] \Phi \\ &= 0. \end{aligned}$$

□

Above lemma indicate that eq.(5.12) is symmetry of D type Drinfeld-Sokolov hierarchy. Using same technique used in [37], following theorem can be derived.

Theorem 5.4. *The flows in eq.(5.12) about additional symmetries of D type Drinfeld-Sokolov hierarchy compose following Block type Lie algebra*

$$[\partial_{c_{m,l}}, \partial_{c_{n,k}}] = (km - nl) \partial_{c_{m+n-1, k+l-1}}, \quad m, n \in \mathbb{Z}_+^{\text{odd}}, k, l \in \mathbb{Z}_+,$$

which holds in the sense of acting on Φ , Φ or \mathcal{L} .

Proof By using eq.(5.12) and eq.(5.13), we get

$$\begin{aligned} [\partial_{c_{m,l}}, \partial_{c_{n,k}}] \Phi &= \partial_{c_{m,l}}(\partial_{c_{n,k}} \Phi) - \partial_{c_{n,k}}(\partial_{c_{m,l}} \Phi) \\ &= -\partial_{c_{m,l}} \left(((\mathcal{M} - \hat{\mathcal{M}})^n \mathcal{L}^k)_- \Phi \right) + \partial_{c_{n,k}} \left(((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_- \Phi \right) \\ &= -(\partial_{c_{m,l}} ((\mathcal{M} - \hat{\mathcal{M}})^n \mathcal{L}^k))_- \Phi - ((\mathcal{M} - \hat{\mathcal{M}})^n \mathcal{L}^k)_- (\partial_{c_{m,l}} \Phi) \\ &\quad + (\partial_{c_{n,k}} ((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l))_- \Phi + ((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_- (\partial_{c_{n,k}} \Phi), \end{aligned}$$

which further leads to

$$\begin{aligned} &[\partial_{c_{m,l}}, \partial_{c_{n,k}}] \Phi \\ &= - \left[\sum_{p=0}^{n-1} (\mathcal{M} - \hat{\mathcal{M}})^p (\partial_{c_{m,l}} (\mathcal{M} - \hat{\mathcal{M}})) (\mathcal{M} - \hat{\mathcal{M}})^{n-p-1} \mathcal{L}^k + (\mathcal{M} - \hat{\mathcal{M}})^n (\partial_{c_{m,l}} \mathcal{L}^k) \right]_- \Phi \\ &\quad - ((\mathcal{M} - \hat{\mathcal{M}})^n \mathcal{L}^k)_- (\partial_{c_{m,l}} \Phi) \\ &\quad + \left[\sum_{p=0}^{m-1} (\mathcal{M} - \hat{\mathcal{M}})^p (\partial_{c_{n,k}} (\mathcal{M} - \hat{\mathcal{M}})) (\mathcal{M} - \hat{\mathcal{M}})^{m-p-1} \mathcal{L}^l + (\mathcal{M} - \hat{\mathcal{M}})^m (\partial_{c_{n,k}} \mathcal{L}^l) \right]_- \Phi \\ &\quad + ((\mathcal{M} - \hat{\mathcal{M}})^m \mathcal{L}^l)_- (\partial_{c_{n,k}} \Phi) \\ &= [(nl - km)(\mathcal{M} - \hat{\mathcal{M}})^{m+n-1} \mathcal{L}^{k+l-1}]_- \Phi \end{aligned}$$

$$= (km - nl)\partial_{c_{m+n-1, k+l-1}}\Phi.$$

Similarly the same results on $\hat{\Phi}$ and \mathcal{L} can be got.

□

Our early papers and above results show the block type algebras are appeared not only in Toda type difference systems but also in differential systems such as two-BKP hierarchy, D type Drinfeld-Sokolov hierarchy, which represents one kind of hidden symmetry algebraic structures of them. These results also show that Block infinite dimensional Lie algebra has a certain of universality in integrable hierarchies.

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